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"The Reciprocity Theorem"

Welcome to the 38th article in the "Circuit Intuitions" column series. As the title suggests, each article provides insights and intuitions into circuit design and analysis. These articles are aimed at undergraduate students but may serve the interests of other readers as well. If you read this article, I would appreciate your comments and feedback, as well as your requests and suggestions for future articles in this series. Please e-mail me your comments at ali@ece.utoronto.ca.

In the previous article in this series [1], we explained Tellegen's theorem and how it relates to the conservation of energy principle. In this article, we explain the reciprocity theorem, another gem in the field of circuit theory, and provide an intuitive understanding for it.

The Reciprocity Theorem

Consider a linear time-invariant two-port network consisting solely of resistors, with no other elements present. Let us number the two ports by 1 and 2 as shown in Figure 1(a). The reciprocity theorem states that if a voltage source is applied to either port 1 or 2 of this network, it will produce the same short circuit current in the other port. That is, $i_{sc2} = i_{sc1}$, as shown in Figure 1(b) and (c). In other words, the *forward* and the *reverse* short circuit transconductances of the two-port network, which we define by $y_f = i_{sc2}/v_0$ and $y_r = i_{sc1}/v_0$, respectively, are identical. The reciprocity theorem is trivial when the two-port network is symmetric, that is, when the two ports can be fully swapped. An example of a symmetric network is where the left half of the network is a mirror image of its right half. Surprisingly, however, the reciprocity theorem holds even when the network is *not* symmetric. One example of this is shown in Figure 2(a), where the network consists of three simple resistors with the values shown, connected in a *T* configuration (also known as a *Y* configuration). This circuit is asymmetric simply because the resistors connected to the left and right ports are different. In fact, because of this asymmetry, the resistance seen from port 1 when port 2 is left open is 25 Ω, whereas the resistance seen from port 2 when port 1 is left open is 20 Ω, implying the two ports do not have the same properties and cannot be swapped. However, one can easily verify that for this circuit, $y_f = y_r = 1/(95\,\Omega)$.

As a second example, consider the circuit shown in Figure 2(b). This circuit consists of five resistors of various values. Again, the circuit is not symmetric, but the reciprocity theorem claims that the forward and reverse transconductances for this circuit are identical. The reader is encouraged to verify this for themselves.

So, why is a resistive network reciprocal, even when the circuit is asymmetric? You are encouraged to pause and ponder this and see whether you can come up with your own intuition. If you find an intuition that is different from what this

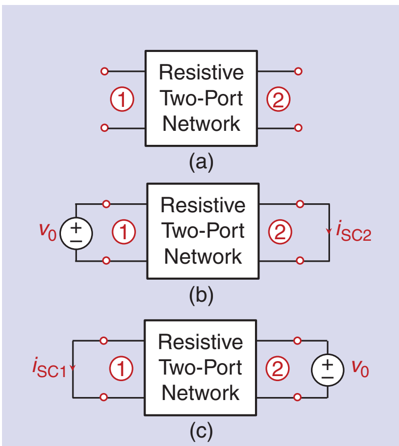

FIGURE 1: (a) A resistive two-port network and a setup to determine its (b) forward and (c) reverse short circuit transconductances.

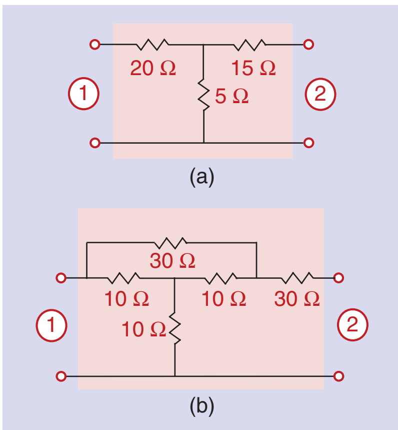

FIGURE 2: (a) An example asymmetric resistive network to test its reciprocity by determining if it has equal forward and reverse transconductances. (b) A second example including five resistors.

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The reciprocity theorem can be proven easily [2] with the help of Tellegen's theorem, covered in the previous article in this series [1]. However, a proof based on Tellegen's theorem might not offer much intuition as to why the reciprocity theorem holds and why it makes sense.

To build an intuition, consider the Π circuit configuration (also known as the Δ configuration) as shown in Figure 3(a), which consists of R_1 and R_2 at ports 1 and 2, respectively, and R_X connecting the two ports. It is easy to see that for this circuit, the short circuit transconductance is $1/R_X$, both in the forward direction, as shown in Figure 3(b), and in reverse, as shown in Figure 3(c). This is because by applying a voltage source to one port and shorting the other, we render R_1 and R_2 ineffective, making the circuit equivalent to a single resistor R_X. Since a single resistor is reciprocal (there is no distinction between the two terminals of a resistor), the Π circuit is intuitively reciprocal.

We now show, through an example, that any resistive circuit can be turned into a Π circuit by a series of $T \rightarrow \Pi\,(Y \rightarrow \Delta)$ and $\Pi \rightarrow T\,(\Delta \rightarrow Y)$ transformations [2] and by replacing any parallel and series resistors with their equivalent resistors.

We have redrawn the circuit of Figure 2(b) in Figure 4(a) to highlight the three resistors that form a T circuit. We then transform this T circuit into a Π circuit as shown in Figure 4(b). Next, we combine the two 30 Ω parallel resistors into a single resistor to form another T circuit on the right (not shown), followed by another T to Π transformation, to arrive at the Π circuit shown in Figure 4(c). We observe that the equivalent resistance connecting the two ports is $R_X = 60\,\Omega$, yielding identical forward and reverse transconductances of $1/(60\,\Omega)$.

To recap, every resistive network has an equivalent Π circuit with a conductance of $1/R_X$ (or a resistance of R_X) connecting the two ports. This unique conductance represents both the forward and the reverse short circuit transconductances of the resistive network. This completes our intuitive proof of the reciprocity theorem.

What we have discussed so far is only one of the three statements associated with the reciprocity theorem [2]. A second statement of this theorem claims that if a current source i_0 is applied to either port 1 or 2 of a resistive network, it will produce the same open circuit voltage at the other port. That is, $v_{OC2} = v_{OC1}$, as shown in Figure 5(a) and (b). In other words, the *forward* and the *reverse* open circuit transimpedances of the two-port network, defined respectively by $z_f = v_{OC2}/i_0$ and $z_r = v_{OC1}/i_0$, are identical.

To see this intuitively, we represent a resistive network by its T-equivalent circuit as shown in Figure 5(c). In this representation, the forward and reverse transimpedances are simply captured by R_X. Notably, when a current source is applied to, say, port 1, while port 2 is kept open, R_1 and R_2 have no impact on the transimpedance; only R_X does.

A third statement of the reciprocity theorem claims that the short circuit current gain in one direction is equal to the open circuit voltage gain in the other direction. Figure 6 captures this claim by showing the setups for the forward short circuit current gain $(h_f = i_{SC2}/i_0)$ and the

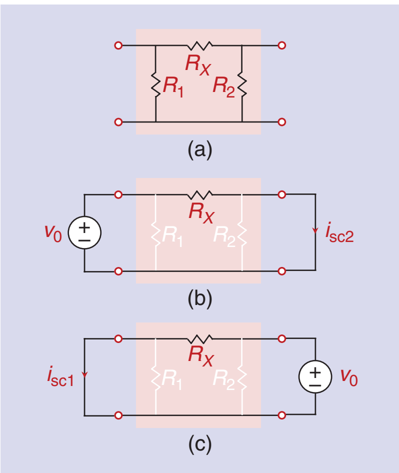

FIGURE 3: (a) A Π circuit represented by three resistors. (b) A setup to determine its forward transconductance will only see R_X as R_1 is in parallel with a voltage source, and R_2 is shorted. (c) A setup to determine its reverse transconductance where R_2 is in parallel with a voltage source and R_1 is shorted. In both cases, the transconductance is determined by R_X only.

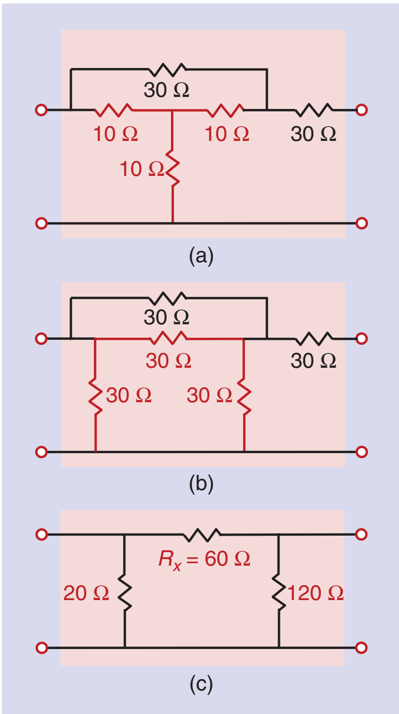

FIGURE 4: Any resistive network such as the one shown in (a) has an equivalent Π circuit as shown in (c), which can be obtained by a sequence of $T \rightarrow \Pi$ [such as from (a) to (b)] and $\Pi \rightarrow T$ transformations and by replacing series and parallel resistors with their equivalent resistors.

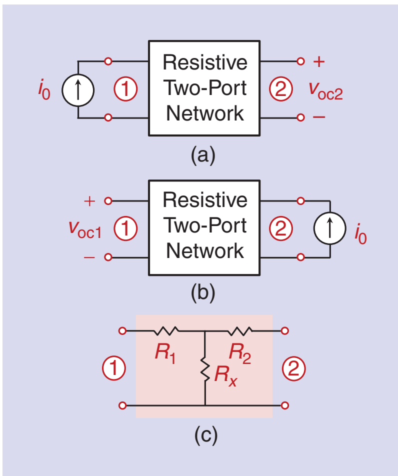

FIGURE 5: A test setup to determine a resistive network's (a) forward and (b) reverse transimpedances. (c) A T-circuit equivalent of the resistive network.

reverse open circuit voltage gain ($h_r = v_{OC1}/v_0$).

Unlike the first two statements, this last statement includes a current gain and a voltage gain, which cannot be represented by a single resistance or conductance. Rather, they need to be represented by a *ratio* of two resistances or two conductances. If we use the equivalent circuit as in Figure 3(a) for our two-port network, we can easily verify that the forward current and the reverse voltage gains are identical and equal to $R_1/(R_1 + R_X)$, whereas the forward voltage and the reverse current gains are identical and equal to $R_2/(R_2 + R_X)$, providing an intuitive understanding of why this last statement of the reciprocity makes sense. Similarly, if we use the *T* equivalent circuit as in Figure 5(c), we can verify that both the forward current and the reserve voltage gains are captured by $R_X/(R_2 + R_X)$, whereas the forward voltage and the reverse current gains are captured by $R_X/(R_1 + R_X)$.

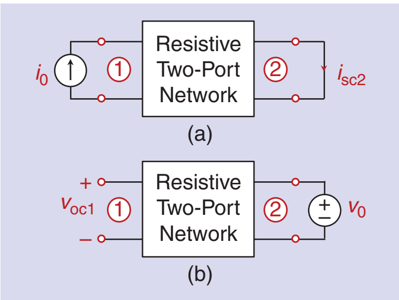

FIGURE 6: A test setup to determine a resistive network's (a) forward current gain (i_{sc2}/i_0) and (b) reverse voltage gain (v_{oc1}/v_0).

Finally, we note that in all three statements of the reciprocity theorem, we rely on representing a resistive network by three parameters only (say the three resistances in a Π or *T* circuit), instead of the four parameters needed in general to model a two-port network. The readers familiar with *y* parameters, *z* parameters, or *h* parameters may recognize that for resistive networks, thanks to reciprocity, $y_{21} = y_{12}, z_{21} = z_{12}$, and $h_{21} = -h_{12}$, reducing the number of parameters needed to model these networks to three.

Before we end this article, we would like to answer the following two questions that may be on your mind: 1) Is the reciprocity theorem valid for all linear-time invariant networks? 2) What is a formal proof for the reciprocity theorem?

1) The reciprocity theorem is valid only for a *subset* of linear time-invariant two-port networks that consist of resistors, inductors, capacitors, coupled inductors, and transformers, and it may not be valid for two-port networks that include dependent sources, independent sources, gyrators, or when a capacitor or an inductor in the circuit has a nonzero initial condition [2]. Also, when the circuit includes capacitors and inductors, the reciprocity theorem must be generalized to use voltage and current phasors (as a function of frequency ω or the Laplace transform parameter s) in its statements.
2) For a comprehensive treatment of the reciprocity theorem and its proof based on Tellegen's theorem, we refer the readers to a classical textbook on circuit theory [2].

In summary, we have shown intuitively that any two-port resistive network is reciprocal in a sense that it exhibits the same forward and reverse short circuit transconductances and the same forward and reverse open circuit transimpedances. Further, we have shown that the open circuit voltage gain in one direction is equal to the short circuit current gain in the other direction. We based our intuitive understanding on the fact that any resistive network can be reduced to both a Π and a *T* circuit, and we demonstrated this through some examples.

References

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EDITOR'S NOTE *(continued from p. 4)*

of luminaries and solid-state circuit techniques and directions.

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Appendix: Related Articles

[A1] D.-K. Jeong, "The beginning of DVI and HDMI: How I untangled the audio/video cables for high-definition displays," *IEEE Solid-State Circuits Mag.*, vol. 16, no. 1, pp. 22–26, Winter 2024, doi: 10.1109/MSSC.2023.3334235.

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[A3] S. Kim, "The early days of Prof. Deog-Kyoon Jeong's lab: Looking back at the 32-year journey with a visionary who reinvented display interfaces," *IEEE Solid-State Circuits Mag.*, vol. 16, no. 1, pp. 31–37, Winter 2024, doi: 10.1109/MSSC.2023.3334242.

[A4] J. Kim, "Daring to put 400 spiral inductors on a chip: Deog-Kyoon Jeong's inspiration and encouragement that made the impossible possible," *IEEE Solid-State Circuits Mag.*, vol. 16, no. 1, pp. 38–41, Winter 2024, doi: 10.1109/MSSC.2023.3334231.

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[A6] B. Razavi, "The design of a biquadratic filter [The Analog Mind]," *IEEE Solid-State Circuits Mag.*, vol. 16, no. 1, pp. 6–13, Winter 2024, doi: 10.1109/MSSC.2023.3336149.

[A7] C. Mangelsdorf, "Accuracy or affectation: Are there too many digits in your work? [Shop Talk: What You Didn't Learn in School]," *IEEE Solid-State Circuits Mag.*, vol. 16, no. 1, pp. 2–5, Winter 2024, doi: 10.1109/MSSC.2023.3336208.

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